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A Solvable Problem in Poisson Statistics

Y. Pomeau[1]

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Among the wide field of interest of Pierre Résibois, the exact solution of various problems in nonequilibrium statistical mechanics took a large place. Recently he used[1] the model of hard rods moving on a line to study some properties in kinetic theory. As a tribute to his memory, I present in this paper the derivation of the exact solution of a problem of Poisson noise.

KEY WORDS: Poisson statistics; time-dependent phenomena; nonlinear random process; one-variable fluctuations.

1. INTRODUCTION

One knows[2] the solution (i.e., any autocorrelation function can be computed in a finite form) of the following class of problems:

$$dz/dt = A\left[F(t)\right]z \tag{1}$$

where z is a p-component real vector, A a $p \times p$ matrix which depends on a real parameter F, and where $F(t)$ is a Poisson stochastic process: $F(t)$ takes a finite number of values (actually I shall limit myself to two values, which I shall call either a/b or $+/-$) and F turns from a to b (or $+$ to $-$) at random with probability $\lambda\, dt$ per time interval dt.

I shall study a (one-dimensional) generalization of (1) to nonlinear situations. The motivation of this work is not purely academic. Actually the dynamics of typical parameters in fluid dynamics near a certain class of bifurcations obey[3] a gradient flow equation of the general form

$$dx/dt = -\partial\psi/\partial x \tag{2}$$

[1] CEN Saclay, Gif-sur-Yvette, France.

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and near the bifurcation, ψ takes the form

$$\psi(x) \simeq -\epsilon x^2/2 + x^4/4 + O(x^6) \tag{3}$$

In the subcritical region, $\epsilon < 0$, the equilibrium $x = 0$ is stable and locally attracting, and $\epsilon > 0$ defines the supercritical domain where $x = 0$ is unstable, the stable equilibria $x = \pm(2\epsilon)^{1/2} + O(\epsilon)^{3/2}$ being locally attracting.

If one adds to (2) a Langevin fluctuating force, i.e., a white Gaussian noise $f(t)$ such that $\langle f(t_1)f(t_2)\rangle = A\,\delta(t_1 - t_2)$, one knows[(4)] that the distribution of the fluctuations of x is given by

$$P(x) = Z^{-1}\exp[-\psi(x)/2A]$$

where Z^{-1} is a normalization constant. With the choice (3) for $\psi(x)$, this probability distribution is perfectly continuous with respect to ϵ. This is due in particular to the Gaussian distribution of the fluctuations of the Langevin force. There is always a [small, as $\exp(-\epsilon/2A)$ as ϵ becomes large] probability that a large fluctuation of this force allows the system to jump from the vicinity of a local minimum of ψ to another minimum. The transition which exists in the absence of fluctuations for the system (2)–(3) is smeared out by this Langevin force. If, on the contrary, one limits the amplitude of the fluctuations of the external noise at a constant level, ϵ independent, the transition between two minima becomes impossible when the potential barrier becomes too large (as ϵ increases). Perhaps this remark is of some importance for the case of metastable thermodynamic situations, for it is unclear that *large*-amplitude fluctuations are Gaussian. Thus the two minima of ψ do not communicate any more if the level of the external noise is bounded, as is the case for a Poisson noise. The case where the external random force takes two values following a Poisson law, ψ being fixed, is a particular case in a large class of one-dimensional problems of the following general form: let $x(t)$ obey an equation of the form

$$dx/dt = \phi(x, F(t))$$

where F is a Poisson process taking two values such that ϕ has two possible forms ϕ_+ and ϕ_-. An elementary extension of the calculations made by Van Kampen[(5)] shows that, if $P_\pm(x,t)\,dx$ is the probability that at time t, ϕ has the form $\phi_\pm$, x being in $[x, x + dx]$, then these two functions obey the coupled system

$$\frac{\partial}{\partial t} P_+(x,t) + \frac{\partial}{\partial x}(\phi_+ P_+) = \lambda(P_- - P_+) \tag{4a}$$

$$\frac{\partial}{\partial t} P_-(x,t) + \frac{\partial}{\partial x}(\phi_- P_-) = \lambda(P_+ - P_-) \tag{4b}$$

where λ is the turning rate of the Poisson process and $\phi_\pm = -(d/dx)\cdot\{\psi_\pm(x)\}$.

2. EQUILIBRIUM DISTRIBUTION

In what follows I shall first solve the system (4) in the stationary (or equilibrium) case.

We are thus looking for a solution of

$$\frac{d}{dx}(\phi_+ P_+^0) = \lambda(P_-^0 - P_+^0)$$

$$\frac{d}{dx}(\phi_- P_-^0) = \lambda(P_+^0 - P_-^0)$$

Adding these two equations, one gets

$$\phi_+ P_+^0 + \phi_- P_-^0 = C, \qquad C \text{ an integration constant}$$

Putting $\chi^0 = P_+^0 - P_-^0$, one has

$$P_\pm^0 = (C \mp \chi^0 \phi_\mp)/(\phi_+ + \phi_-)$$

and

$$\frac{1}{2}\frac{\partial}{\partial x}\left(C\frac{\phi_+ - \phi_-}{\phi_+ + \phi_-}\right) + \frac{\partial}{\partial x}(\Lambda\chi^0) + \lambda\chi^0 = 0 \tag{5}$$

where

$$\Lambda(x) = \frac{\phi_+(x)\phi_-(x)}{\phi_+(x) + \phi_-(x)}$$

Due to the physical origin of the problem, I assume that the support of P_+^0 and P_-^0 and of χ^0 is made of one of the segments $[x_+, x_-]$ (or eventually $[x_-, x_+]$) such that x_+ and x_- are stable equilibria of ψ_+ and ψ_- (or zeros of the velocity fields ϕ_+ and ϕ_-) and that no other stable equilibrium of ψ_+ or ψ_- lies in this interval. During its motion the particle will move toward x_+ or x_-, depending upon whether the potential has the form ψ_+ or ψ_-, but it will never pass these stable equilibria. These equilibria are indeed zeros of $\Lambda(x)$. Due to (5), χ^0 and Λ must vanish together at x_- and x_+, which implies $C = 0$, and the general solution of (5) with $C = 0$ is

$$\chi^0 = \frac{K}{\Lambda(x)}\exp\left[-\int_{x_+}^{x}\frac{\lambda\, dy}{\Lambda(y)}\right]$$

and

$$P_\pm^0 = \mp\frac{K}{\phi_\pm(x)}\exp\left[-\int_{x_+}^{x}\frac{\lambda\, dy}{\Lambda(y)}\right]$$

It is easy to see that, with a convenient choice of the sign of the normalizing factor K, $P^0_\pm$ are indeed two positive functions in the interval $[x_+, x_-]$: by assumption, $\phi_\pm$ are two velocity fields without other zeros than $x_\pm$ in $[x_+, x_-]$.

The choice of K is determined by the normalization condition

$$\int_{x_+}^{x_-} dx \left[P^0_+(x) + P^0_-(x) \right] = 1$$

or

$$K^{-1} = \int_{x_+}^{x_-} dx \, \frac{\phi_- - \phi_+}{\phi_- \phi_+}$$

The form of this condition leads us to consider the convergence of the normalizing integral near the zeros of ϕ_+ and ϕ_- (and of Λ). The discussion made below for x_+ extends in an obvious way to the other cases.

In the generic case of a parabolic minimum of the potential, one has

$$\phi_+ \underset{x \to x_+}{\simeq} -\alpha(x - x_+) \qquad \text{with} \quad \alpha > 0$$

and

$$\Lambda \underset{x \to x_+}{\simeq} -\alpha(x - x_+)$$

Thus

$$\frac{\phi_- - \phi_+}{\phi_- \phi_+} \exp\left[-\int_{x_+}^{x} \frac{\lambda \, dy}{\Lambda(y)} \right] \underset{x \to x_+}{\simeq} -\frac{1}{\alpha(x - x_+)^{1 - \lambda/\alpha}}$$

As $\lambda/\alpha > 0$, the normalization integral converges at $x = x_+|_+$.

If ϕ_+ has a power law behavior near $x = x_+|_+$,

$$\phi \underset{x \to x_+}{\simeq} -\alpha|x - x_+|^\rho, \qquad \alpha, \rho > 0$$

Thus

$$\Lambda \underset{x \to x_+}{\simeq} -\alpha|x - x_+|^\rho$$

and

$$\frac{\phi_- - \phi_+}{\phi_+ \phi_-} \exp\left(-\int_{x_+}^{x} \frac{\lambda \, dy}{\Lambda} \right) \underset{x \to x_+}{\simeq} -\frac{1}{\alpha|x - x_+|^\rho} \exp\left(\frac{\lambda}{\alpha} \, \frac{|x - x_+|^{1-\rho}}{1 - \rho} \right)$$

$$\underset{0 < \rho < 1}{\simeq} -\frac{1}{\alpha|x - x_+|^\rho}$$

$$\underset{1 < \rho}{\simeq} \exp\left(\frac{\lambda}{\alpha} \, \frac{|x - x_+|^{1-\rho}}{1 - \rho} \right)$$

In all these cases, the normalization integral converges. In a particular case one may obtain a simple explicit solution:

$$\phi_\pm(x) = -x \pm 1, \qquad x_\pm = \mp 1, \qquad \Lambda = \frac{1-x^2}{2x}$$

$$\chi^0 = \frac{2xK}{1-x^2}\exp\left(-\lambda\int_{-1}^{x}\frac{2y\,dy}{1-y^2}\right) = \frac{2xK}{(1-x^2)^{1-\lambda}}$$

$$P_\pm^0 = \mp\frac{K}{-x\pm 1}(1-x^2)^\lambda$$

with

$$K^{-1} = \int_{-1}^{+1} 2\,dx\,(1-x^2)^{\lambda-1} = 2B\left(\frac{1}{2},\lambda\right) = \frac{\Gamma(1/2)\Gamma(\lambda)}{\Gamma(\lambda+1/2)}$$

3. TRANSITION IN THE EQUILIBRIUM DISTRIBUTION

Let us look at the change in the equilibrium distribution near a transition. We assume in this section that ϕ_+ or ϕ_- (say ϕ_+) depends continuously on a parameter and that, when this parameter crosses a critical value, a new equilibrium for ψ_+ appears in the interval $]x_+, x_-[$. This bifurcation is schematically pictured in the Fig. 1. It is obvious from this figure that after the bifurcation, the motion stays in the interval $[x'_+, x_-]$, x'_+ being the new equilibrium of ψ_+.

In the vicinity of this new equilibrium, the generic form of $\phi_+(x)$ is $\phi_+(x) = \eta - \beta(x-x_S)^2$, with $\beta > 0$. For $\eta = 0$, x_S is a metastable equilibrium of ψ_+. For $\eta > 0$, two equilibria appear in the vicinity of x_S, at the zeros of ϕ_+, i.e., at $x = x_S \pm (\eta/\beta)^{1/2}$. Of course the point $x'_+ = x_S + (\eta/\beta)^{1/2}$ is the new stable equilibrium, although $x_S - (\eta/\beta)^{1/2}$ is unstable.

For $\eta > 0$, the support of the equilibrium distribution is $[x'_+, x_S]$. Let us look at the transition to this situation. For a slightly negative η, the function Λ varies rapidly in a region of extent $(|\eta|/\beta)^{1/2}$ around x_S. In this region

$$\frac{1}{\Lambda} \simeq \frac{1}{\phi_+} \simeq \frac{-1}{|\eta| + \beta(x-x_S)^2}$$

and the corresponding contribution to

$$\int\frac{dy\,\lambda}{\Lambda}$$

is

$$\lambda\int_{y\approx x_S}\frac{dy\,\lambda}{\Lambda} \simeq -\lambda\int_{y\approx x_S}\frac{dy}{|\eta| + \beta(y-x_S)^2} \simeq -\frac{\lambda\pi}{(\beta|\eta|)^{1/2}}$$

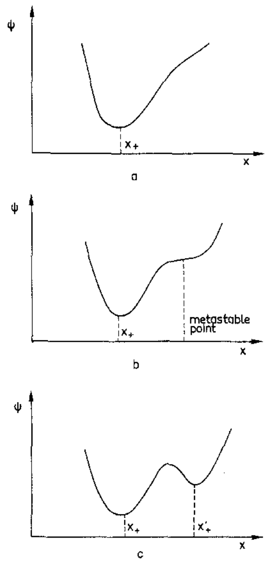

Fig. 1. Deformation of the potential $\psi_+(x)$ near a bifurcation. In (a) a single equilibrium exists, in (b), that is just at the bifurcation, a metastable equilibrium exists that produces in (c) a new equilibrium position in x'_+.

Thus, as $\eta \to 0_-$, the amplitude of χ^0 (and of $P^0_\pm$) in the region $x_+ < x < x_S$ is of order $\exp[-\lambda\pi/(\beta|\eta|)^{1/2}]$ with respect to its value in the region $x_S < x < x_-$. Of course this estimation is not valid in the transition region of width $(|\eta|/\beta)^{1/2}$ around x_S.

The reason for this situation is as follows: as η goes to 0_-, x_S is very near to being a metastable point, and the trajectories running from x_- to x_+ take a very long time to run over x_S when the potential is ψ_+, although when the potential is in the state ψ_- the particle runs at a finite velocity over x_S toward x_-, since x_S is not at all a particular point for ψ_-.

In the limit situation ($\eta = 0$), the support of the equilibrium measure is $[x'_+ = x_S, x_-]$ and the density of probability is exponentially small, as $\exp[-\lambda/\beta(x - x_S)]$ near $x_S|_+$.

4. NONEQUILIBRIUM PROBABILITY DISTRIBUTION

It does not seem possible to get in a closed form the solution of the relaxation problem [that is, the solution of (4) at time t, for given initial conditions]. I shall give in what follows only partial results on this problem.

Let s be an eigenvalue of the relaxation problem and let $Q_\pm$ be the corresponding pair of eigenfunctions. They satisfy the system of coupled equations

$$sQ_+ + \partial_x(\phi_+ Q_+) = \lambda(Q_- - Q_+) \tag{6a}$$

$$sQ_- + \partial_x(\phi_- Q_-) = \lambda(Q_+ - Q_-) \tag{6b}$$

It is easy to obtain from (6) a single equation for either Q_+ or Q_-. I shall, however, try to respect as much as possible the $+/-$ symmetry, keeping clear at each step of the calculation the existence of an elementary solution of (6) with $s = 0$.

Taking as new functions $q_\pm(x) = e^{-\alpha(x)}Q_\pm(x), \alpha(x)$ arbitrary (for the moment), one has

$$(s + \lambda + \phi_\pm \alpha_x)q_\pm + \frac{\partial}{\partial x}(\phi_\pm q_\pm) = \lambda q_\mp$$

where $(\cdot)_x = (\partial/\partial x)(\cdot)$.

Putting $M = \phi_+ q_+ + \phi_- q_-$, $L = \phi_+ q_+ - \phi_- q_-$, so that $q_\pm = (M \pm L)/2\phi_\pm$, one has

$$M\left(\alpha_x + \frac{s\Lambda}{2}\right) + \frac{Ls}{2\phi_+\phi_-}(\phi_- - \phi_+) + M_x = 0 \tag{7a}$$

$$\left(\lambda + \frac{s}{2}\right)M\left(\frac{1}{\phi_+} - \frac{1}{\phi_-}\right) + L\left[\alpha_x + \left(\frac{s}{2} + \lambda\right)\right] + L_x = 0 \tag{7b}$$

where $\Lambda = (\phi_+ + \phi_-)/\phi_+\phi_-$, as before.

By a convenient choice of α (and thus of α_x) one may either eliminate the term proportional to M in (7a) or to L in (7b), allowing one to find L (or M) as a function of M_x (or L_x). Choosing the first alternative, i.e., $\alpha_x = -s\Lambda/2$, I get

$$L = \frac{2\phi_+\phi_-}{s(\phi_+ - \phi_-)} M_x$$

and

$$\frac{s}{2}\left(\lambda + \frac{s}{2}\right)M\left(\frac{1}{\phi_+} - \frac{1}{\phi_-}\right) + \frac{\lambda\Lambda\phi_+\phi_-}{\phi_+ - \phi_-}M_x + \left(\frac{\phi_+\phi_-}{\phi_+ - \phi_-}M_x\right)_x = 0 \tag{8}$$

The next step consists in eliminating the first-order derivative M_x in (8). This can be done in many different ways. I follow a method making always explicit the existence of a solution with $s = 0$.

Putting

$$a(x) = \frac{\phi_+\phi_-}{\phi_+ - \phi_-} \qquad \text{and} \qquad b(x) = \frac{\lambda\Lambda}{\phi_+ - \phi_-}$$

I introduce a new variable $X(x)$ such that

$$X_x(a_x + b) + aX_{xx} = 0 \tag{9}$$

or

$$X = \int^x dy \exp\left(- \int^y dz \frac{a_z + b}{a} \right) \tag{10}$$

It is remarkable that X does not depend on s. It is legitimate to take X as a new variable instead of x, since X is a strictly increasing function of x. After substitution of this variable in (8), one gets

$$M_{XX} + k(s)U(X)M = 0 \tag{11}$$

where $k(s) = (s/2 + \lambda)s/2$, X being defined implicitly as a function of x by (10) and where $U(X) = -(\phi_+ - \phi_-)^2/(X_x^2\phi_+^2\,\phi_-^2)$. On this (final) form I make the following remarks:

(i) $U(X)$ is infinite at $X_\pm$, images of $x_\pm$ in the variable X [X being defined by (10), $X_\pm \equiv X(x_\pm)$]. If one *assumes* that M is in $L^2[X_+, X_-]$ (this is *not at all* a necessary property, the square of a probability distribution has no physical meaning), the operator $\partial^2_{X^2}$ is negative as $-U(X)$, thus $k(s)$ must be real negative, and either s is real negative with $-2\lambda \leqslant s \leqslant 0$ or s has an imaginary part and is of the form $s = -\lambda + is''$, s'' real.

(ii) The eigenfunctions of (11) are defined by pairs: if s is an eigenvalue of (6a), $(-s - 2\lambda)$ is an eigenvalue, too, as the substitution $s \to (-s - 2\lambda)$ leaves $k(s)$ invariant. If s has an imaginary part, the conjugated eigenvalue is precisely given by this substitution.

As an application, consider the equilibrium solution, for which $s = 0$. From the above substitutions we deduce the existence of a relaxation eigenfunction with the eigenvalue -2λ. It has the form

$$M = M^0 \exp\left[-\lambda \int^x \Lambda(y)\, dy \right], \qquad L = L^0 \exp\left[-\lambda \int^x \Lambda(y)\, dy \right], \ldots$$

where M^0 and L^0 are the equilibrium values of M and L. This can be verified by elementary substitutions in (7) with $s = -2\lambda$ and $\alpha_x = -\lambda\Lambda$.

(iii) It is difficult to find directly the behavior of the relaxation rate in the vicinity of a "transition" (in the sense of Section 3). However, a simple argument shows that near a transition the first "excited" mode of (11) has an eigenfrequency which vanishes exponentially.

Near a transition (as defined in Section 3), a quasimetastable point for ϕ_+ (for instance) appears in the interval $[x_+, x_-]$, say at x_S. A very large relaxation time appears under the same conditions. This is approximately the time needed for the particle to move from x_- to x_+ across x_S. A rough estimate of this time is given by the ratio of the probabilities of presence (at equilibrium) in $[x_S, x_+]$ to the one in $[x_-, x_S]$. Near the transition, this ratio is of order $\exp[-\lambda\pi/(\beta|\eta|)^{1/2}]$ (β, η have the same definitions as in Section 3). Thus, it is natural to conclude that near $\eta = 0_-$ one of the eigenvalues of (6) goes to zero as $\exp[-\lambda\pi/(\beta|\eta|)^{1/2}]$. Such a result could be likely derived from a Kramers-like analysis of the relaxation problem.

For the symmetric double-well potential of the introduction, the onset of transition manifests itself by a breaking of the symmetry $x/-x$. The typical correlation time diverges at this transition as $\exp(-C/|\eta|^{1/2})$.

5. CONCLUSION

This solution of a problem of one-dimensional Poisson statistics has its own interest. A natural question arises about the possibility of an extension to higher dimensions. This is an extremely difficult problem; it is even unclear if the resulting equilibrium distribution is smooth or some more or less pathological function, despite the fact that the equations equivalent to (4) can be written at once for any kind of velocity field in any number of dimensions. One can see again on this model how much the existence of solutions is often limited to one-dimensional problems, which are known to be rather misleading sometimes!

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